Helmut's cookbook recipe number 4:

Vector Calculation in Index Notation (Einstein's Summation Convention)

(Version 10, 8.3.2025)

This "cookbook recipe" explains how to convert vector calculations into index notation according to Einstein's summation convention, perform a calculation in this index notation, and back-convert the final result into vector notation. Here, a Euclidean metric is assumed (e.g. calculations in $\mathbb{R}^3$), and therefore no distinction is made between co- and contravariant indices.

1 General information

1.1 Fundamentals

In conventional vector notation, one writes a vector with n components as follows:

$$\vec{v} = \begin{pmatrix} v_1 \\ \cdots \\ v_n \end{pmatrix} \tag{1}$$

In index notation, you simply write v_i, where the index i stands for the i-th vector component of $\vec{v}$:

$$(\vec{v})_i = v_i \tag{2}$$

If you write v_i this stands for a single, selected component of the vector (namely the i-th). Thus v_i initially represents a scalar value (a number). On the other hand, you can replace i with any number between 1 and n and thus v_i also represents the entire vector $\vec{v}$ (instead of i, any other letter can of course also be used as an index). **A term with a single index (the so-called "free index") therefore represents a vector in index notation.**

1.2 Summation Convention

If the same index occurs exactly twice in a product term (if it is "saturated"), it implies summation of that term over all the values of the index ("Einstein's summation convention"). For example:

$$a_i v_i \equiv \sum_{i=1}^{n} a_i v_i = a_1 v_1 + \cdots + a_n v_n \tag{3}$$

The example above corresponds to the inner product of two vectors.

$$(\vec{a} \cdot \vec{v})_i = a_i v_i \tag{4}$$

Two variables with the same index in the same product term ("saturated index") therefore represent an inner vector product and thus a scalar.

Rules:

- The same index may never appear more than twice in a product term
- A fraction is also a product term!
- As a beginner, perhaps do not write a_i^2, but $a_i a_i$, so that you recognize that you are adding over i .
- $\sqrt{a_i a_i}$ means $\sqrt{\sum_{i=1}^{n} a_i a_i}$, not $\sum_{i=1}^{n} \sqrt{a_i a_i}$
- Big simplification: Because, for example a_i, b_i and c_j represent the i-th or, respectively, the j-th component of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (i.e. they are scalars), their order in a product term does not matter at all: $a_i b_i c_j = c_j a_i b_i = b_i c_j a_i = b_i a_i c_j = c_j b_i a_i$

1.3 Free index and saturated indices

If there are only double ("saturated") indices in a product term, then this term represents a scalar expression. Example: $a_i b_i c_j d_j$. Note that, if this is the case in one term, then all product terms that are connected to this term additively or subtractively must have only saturated indices (because, an addition or subtraction of a vector and a scalar is not possible).

Conversely: If there is a "free" index in a product term (i.e. an index that only occurs once), then this product term represents a vector. The left and right sides of the equation, as well as all occurring terms, must have the same free index.

Correct:

$$c_j = a_i u_i v_j + b_k v_k v_j \text{ (vector)} \tag{5}$$

$$\alpha = a_i u_i + b_j v_j \text{ (scalar)} \tag{6}$$

Incorrect:

$$c_j = a_i u_i v_j + b_i v_i v_k \tag{7}$$

$$c_j = a_i u_i + b_i v_i v_j \tag{8}$$

$$\alpha = a_i b_i c_i + b_i v_i v_j \tag{9}$$

- Mistake in (7): The free index k on the right side does not match the free index j on the left side of the equation.
- Mistake in (8): On the left side of the equation there is a vector (free index j), but the term $a_i u_i$ is a scalar.
- Mistake in (9): The term $a_i b_i c_i$ contains the same index three times, and there is also a scalar on the left-hand side of the equation, and the right-hand expression $b_i v_i v_j$ with free index j represents a vector.

Also note that the designation of the saturated indices can be chosen arbitrarily, e.g:

$$c_j = a_i u_i v_j = a_j u_j v_j = a_k u_k v_j = \cdots \tag{10}$$

$$\alpha = a_i c_i u_j v_j = a_j c_j u_i v_i = a_m c_m u_n v_n = \cdots \tag{11}$$

2 Special Symbols

2.1 Position Vector

We want to use the convention here that we always use the letter x for the position vector, i.e. in vector notation

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{12}$$

or in index notation

$$(\vec{x})_i = x_i, \tag{13}$$

whereby x_1 stands for x, x_2 stands for y, and x_3 stands for z. In contrast, all other letters stand for vector fields in which each component can (potentially) depend on all three spatial directions (and also on the time t), e.g:

$$\vec{v} = \begin{pmatrix} v_1(x,y,z,t) \\ v_2(x,y,z,t) \\ v_3(x,y,z,t) \end{pmatrix} \triangleq v_i \tag{14}$$

2.2 Spatial Derivative

$$\vec{\nabla}_i = \partial_i \stackrel{\text{def}}{=} \frac{\partial}{\partial x_i} \Leftrightarrow \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z} \tag{15}$$

Note: We follow the convention that the differential operator only acts on the object immediately to the right of it. Therefore: Always use parentheses if the operator acts on several objects, and if you prefer so, also otherwise for clarity. For example, the product rule is

$$\partial_i(u_i v_j) = (\partial_i u_i) v_j + u_i(\partial_i v_j) \equiv \partial_i u_i v_j + u_i \partial_i v_j \tag{16}$$

2.3 Time Derivative

$$\partial_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t} \tag{17}$$

To avoid confusion, it is advisable not to use the letter t as an index.

2.4 Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \tag{18}$$

2.5 Levi Civita Symbol

$$\varepsilon_{ijk\ldots} = \begin{cases} 1, \text{if } (ijk\ldots) \text{ is an even permutation of } (123\ldots) \\ -1, \text{if} (ijk\ldots) \text{ is an odd permutation of } (123\ldots) \\ 0 \text{ otherwise (i.e. if at least two indices are identical)} \end{cases} \quad (19)$$

Examples:

- $\varepsilon_{123} = 1$ (according to definition)
- $\varepsilon_{132} = -1$ (because a single permutation is necessary: $\varepsilon_{123} \rightarrow \varepsilon_{132}$)
- $\varepsilon_{312} = 1$ (because two permutations are necessary: $\varepsilon_{123} \rightarrow \varepsilon_{132} \rightarrow \varepsilon_{312}$)
- $\varepsilon_{122} = 0$ (because the index 2 occurs twice)

3 Index Notation When Applying the Chain Rule

Vector Notation	Equivalent Index Notation	Free Index
$\vec{\nabla}\,\mathrm{f}(\mathrm{g}(x,y,z)) = \frac{\partial \mathrm{f}}{\partial \mathrm{g}} \vec{\nabla}\mathrm{g}(x,y,z)$	$\partial_i\,\mathrm{f}(\mathrm{g}) = \mathrm{f}'(\mathrm{g})\,\partial_i \mathrm{g}$	i
$\frac{\partial}{\partial t}\vec{v}(\vec{w}(x,y,z,t)) = \underbrace{D\vec{v}(\vec{w})}_{\text{Fréchet-derivative}} \cdot \frac{\partial}{\partial t}\vec{w}(t)$	$\partial_t\, v_i(\vec{w}) = \partial_j v_i \partial_t w_j$	i

4 Conversion Between Vector and Index Notation

Rule	Object/Operation	Vector Notation	Equivalent Index Notation	Free Index
(a)	Vector	$\vec{v}$	v_i	i
(b)	Scalar Product	$\vec{u} \cdot \vec{v}$	$u_i v_i$	-
(c)	Magnitude Squared	$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$	$v_i v_i$	-
(d)	Vector Length	$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$	$\sqrt{v_i v_i}$	-
(e)	Gradient (scalar)	$\vec{\nabla}\,\mathrm{f}(x,y,z)$	$\partial_i \mathrm{f}$	i
(f)	Divergence	$\vec{\nabla} \cdot \vec{v}$	$\partial_i v_i$	-
(g)	Cross Product	$\vec{\mathrm{u}} \times \vec{v}$	$\varepsilon_{ijk} u_j v_k$	i
(h)	Rotation	$\vec{\nabla} \times \vec{v}$	$\varepsilon_{ijk} \partial_j v_k$	i
(i)	Gradient (Vectorial)	$\mathrm{grad}(\vec{v}) \equiv \vec{\nabla} \otimes \vec{v} \equiv \vec{\nabla}\vec{v}$	$(\vec{\nabla}\vec{v})_{ij} \triangleq \partial_i v_j$	i, j
(j)	Laplacian (Scalar)	$\vec{\nabla}^2\,\mathrm{f}(x,y,z) = \vec{\nabla} \cdot (\vec{\nabla}\mathrm{f})$	$\partial_i \partial_i f$	i
(k)	Laplacian (Vectorial)	$\vec{\nabla}^2 \vec{v} = \begin{pmatrix} \vec{\nabla}^2 v_1 \\ \vec{\nabla}^2 v_2 \\ \vec{\nabla}^2 v_3 \end{pmatrix}$	$\partial_j \partial_j v_i$	i
(l)	Directional Derivative of $\vec{v}$ along $\vec{u}$	$(\vec{u} \cdot \nabla)\vec{v}$	$u_j \partial_j v_i$	i
(m)	Matrix-Vector Multiplication	$\underline{\underline{\mathrm{M}}}\vec{v}$	$\mathrm{M}_{ij} v_j$	i

Note regarding point (i) (vectorial gradient): The result of this operation is (in conventional vector notation) a tensor of level two, which can be written as a matrix. The index i represents the rows, and the index j the columns of the matrix. Also in point (m), index i represents the rows and index j the columns of matrix $\underline{\underline{\mathrm{M}}}$.

5 Calculation Rules and Transformations

$$\delta_{ij} = \delta_{ji} \tag{20}$$

$$\delta_{ii} = n \tag{21}$$

$$\delta_{ij}\delta_{jk} = \delta_{ik} \tag{22}$$

$$\delta_{ij}a_i = a_j \tag{23}$$

$$\delta_{ij}a_j = a_i \tag{24}$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \det\begin{pmatrix}\delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm}\end{pmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{25}$$

$$\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl} \tag{26}$$

$$\varepsilon_{ij}\varepsilon_{ij} = 2!;\ \varepsilon_{ijk}\varepsilon_{ijk} = 3!;\ \varepsilon_{ijkl}\varepsilon_{ijkl} = 4!, \ldots \tag{27}$$

$$\varepsilon_{ijk}\delta_{ij} = 0;\ \varepsilon_{ijk}\delta_{ik} = 0;\ \varepsilon_{ijk}\delta_{jk} = 0 \tag{28}$$

$$\partial_i x_j = \delta_{ij} \tag{29}$$

$$\partial_i x_i = \delta_{ii} = n \tag{30}$$

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} \tag{31}$$

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} \tag{32}$$

$$\varepsilon_{ijk}\partial_i\partial_j a_k = \varepsilon_{ijk}\partial_i\partial_k a_j = \varepsilon_{ijk}\partial_j\partial_k a_i = 0 \tag{33}$$

Notes regarding calculation rule (25):

- If the index naming is different, you can of course rename the indices accordingly for clarity (but then of course in the whole equation for the free index)
- If the index sequence is different, the indices can be combined taking into account rule (19) until you reach the representation (25).

Notes on the calculation rules (21) and (33):

- The letter n stands for the number of dimensions (for calculations in $\mathbb{R}^3$ it is therefore $n = 3$).

6 Strategy

Here is a strategy for conversion and calculation:

1. First, consider the given equation in vector notation. Which parts represent scalars? Vector products or gradients are scalars, for example. Which parts are vector-valued? Is the total expression a vector or a scalar?
2. Now use the rules from chapter 4 to convert the expression. Proceed step by step and assign the free index to the sub-expression that makes the overall expression a vector.

   Example: Given, for example, the expression (34)

   $$\vec{v} = (\vec{a}\cdot\vec{b})\vec{c} \tag{34}$$

   $(\vec{a}\cdot\vec{b})$ is scalar, only through $\vec{c}$ the total expression becomes vectorial again. Let's assign the free index and call it i.

   $$(\vec{v})_i = (\vec{a}\cdot\vec{b})(\vec{c})_i \tag{35}$$

   In index notation:

   $$v_i = (\vec{a}\cdot\vec{b})c_i \tag{36}$$

We convert the vector $\vec{v}$ using rule (a) from chapter 4 and the scalar product $\vec{a} \cdot \vec{b}$ using rule (b) from chapter 4 where we make sure that we have already used the index i has already been used:

$$v_i = a_j b_j c_i \tag{37}$$

3. The converted equation can now be calculated using the calculation rules from chapter 5 to transform it. Often you first try to use rules (25), (26), (27) and (28) to either get rid of the Levi-Civita symbols or to convert them into Delta symbols, which can then be converted using the rules (20), (21), (22), (23) and (24) are "contracted" with the variables.
4. For derivatives, strictly observe the product or quotient rule. Important: Don't forget the parentheses that indicate what a particular differential operator acts on! Then rules (29) and (33) apply.
5. If an index occurs more than twice when multiplying out brackets, always assign new index names.
6. Do not forget: In a product term, the variables can be arranged in any order. $a_j b_j c_i$ is the same as $a_j c_i b_j$. Both represent the vector expression $(\vec{a} \cdot \vec{b})\vec{c}$.
7. With the help of the rules from chapter 4 the equation can be converted back to vector notation.

7 Example

The following equation is to be transformed (where $\vec{q}$ is a constant vector):

$$\vec{v} = \vec{\nabla} \times \frac{\vec{q} \times \vec{x}}{\|\vec{x}\|} \tag{38}$$

Obviously there is a vector on both sides of the equation. We assign the free index i and use rule (h) from chapter 4 to convert the first cross product:

$$v_i = \varepsilon_{ijk} \partial_j \left(\frac{\vec{q} \times \vec{x}}{\|\vec{x}\|} \right)_k \tag{39}$$

We continue with the cross product in the numerator. Here, the free index of the partial expression $\vec{q} \times \vec{x}$ (which is no longer free in the total term due to ε_{ijk}) according to equation (39) is already determined by k :

$$v_i = \varepsilon_{ijk} \partial_j \frac{\varepsilon_{klm} q_l x_m}{\|\vec{x}\|} \tag{40}$$

Finally, we can also substitute $\|\vec{x}\|$ by using rule (d) from chapter 4, where we note that the indices i, j, k, l and m are already assigned:

$$v_i = \varepsilon_{ijk} \partial_j \frac{\varepsilon_{klm} q_l x_m}{\sqrt{x_n x_n}} \tag{41}$$

ε_{klm} and q_i are constant. We can therefore place them in front of the differential operator ∂_j. We can also express $\frac{1}{\sqrt{x_n x_n}}$ as $(x_n x_n)^{-1/2}$:

$$v_i = q_l \varepsilon_{ijk} \varepsilon_{klm} \partial_j \left[x_m (x_n x_n)^{-\frac{1}{2}} \right] \tag{42}$$

We apply the product rule:

$$v_i = q_l \varepsilon_{ijk} \varepsilon_{klm} \left[\partial_j (x_m)(x_n x_n)^{-\frac{1}{2}} + x_m \partial_j (x_n x_n)^{-\frac{1}{2}} \right] \tag{43}$$

According to rule (29) we can substitute $\partial_j(x_m) = \delta_{jm}$

$$v_i = q_l \varepsilon_{ijk} \varepsilon_{klm} \left[\delta_{jm} (x_n x_n)^{-\frac{1}{2}} + x_m \partial_j (x_n x_n)^{-\frac{1}{2}} \right] \tag{44}$$

Side calculation of $\partial_j (x_n x_n)^{-\frac{1}{2}}$:

We apply the chain rule:

$$\partial_j (x_n x_n)^{-\frac{1}{2}} = -\frac{1}{2}(x_n x_n)^{-\frac{3}{2}} \partial_j (x_n x_n) \tag{45}$$

On $\partial_j (x_n x_n)^{-\frac{1}{2}}$ we apply the product rule:

$$\partial_j (x_n x_n)^{-\frac{1}{2}} = -\frac{1}{2}(x_n x_n)^{-\frac{3}{2}} \left[\partial_j (x_n) x_n + x_n \partial_j (x_n)\right] \tag{46}$$

Of course $\partial_j (x_n) x_n + x_n \partial_j (x_n) = 2x_n \partial_j x_n$:

$$\partial_j (x_n x_n)^{-\frac{1}{2}} = -\frac{1}{2}(x_n x_n)^{-\frac{3}{2}} 2x_n \partial_j x_n \tag{47}$$

According to rule (29) $\partial_j x_n = \delta_{jn}$. Additionally, ½ cancels out with 2:

$$\partial_j (x_n x_n)^{-\frac{1}{2}} = -(x_n x_n)^{-\frac{3}{2}} x_n \delta_{jn} \tag{48}$$

According to rule (24) $x_n \delta_{jn} = x_j$:

$$\partial_j (x_n x_n)^{-\frac{1}{2}} = -x_j (x_n x_n)^{-\frac{3}{2}} \tag{49}$$

End of side calculation

We now substitute the result of the side calculation (49) into equation (44):

$$v_i = q_l \varepsilon_{ijk} \varepsilon_{klm} \left[\delta_{jm} (x_n x_n)^{-\frac{1}{2}} - x_m x_j (x_n x_n)^{-\frac{3}{2}}\right] \tag{50}$$

Of course, this can also be written like this:

$$v_i = q_l \varepsilon_{ijk} \varepsilon_{klm} \left[\frac{\delta_{jm}}{\sqrt{x_n x_n}} - \frac{x_m x_j}{(x_n x_n)^{3/2}}\right] \tag{51}$$

We multiply q_l into the bracket:

$$v_i = \varepsilon_{ijk} \varepsilon_{klm} \left[\frac{q_l \delta_{jm}}{\sqrt{x_n x_n}} - \frac{q_l x_m x_j}{(x_n x_n)^{3/2}}\right] \tag{52}$$

We convert $\varepsilon_{ijk} \varepsilon_{klm}$ by means of rule (25):

$$v_i = \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right) \left[\frac{q_l \delta_{jm}}{\sqrt{x_n x_n}} - \frac{q_l x_m x_j}{(x_n x_n)^{3/2}}\right] \tag{53}$$

Expanding the terms results into:

$$v_i = \frac{q_l \delta_{jm} \delta_{il} \delta_{jm}}{\sqrt{x_n x_n}} - \frac{q_l \delta_{jm} \delta_{im} \delta_{jl}}{\sqrt{x_n x_n}} - \frac{q_l x_m x_j \delta_{il} \delta_{jm}}{(x_n x_n)^{\frac{3}{2}}} + \frac{q_l x_m x_j \delta_{im} \delta_{jl}}{(x_n x_n)^{3/2}} \tag{54}$$

The numerator of the first fraction contains $\delta_{jm} \delta_{jm}$. According to rule (20) this is equal to $\delta_{jm} \delta_{mj}$. According to rule (22) this contracts to δ_{jj} and according to rule (21) is equal to the dimension n (here: 3). Therefore $\delta_{jm} \delta_{jm} = \delta_{jj} = 3$, and therefore :

$$v_i = \frac{3 q_l \delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l \delta_{jm} \delta_{im} \delta_{jl}}{\sqrt{x_n x_n}} - \frac{q_l x_m x_j \delta_{il} \delta_{jm}}{(x_n x_n)^{3/2}} + \frac{q_l x_m x_j \delta_{im} \delta_{jl}}{(x_n x_n)^{3/2}} \tag{55}$$

The numerator of the second fraction contains $\delta_{jm} \delta_{im}$. According to rule (20) this is the same as $\delta_{jm} \delta_{mi}$. According to rule (22) this contracts to δ_{ji}:

$$v_i = \frac{3 q_l \delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l \delta_{jl} \delta_{ji}}{\sqrt{x_n x_n}} - \frac{q_l x_m x_j \delta_{il} \delta_{jm}}{(x_n x_n)^{3/2}} + \frac{q_l x_m x_j \delta_{im} \delta_{jl}}{(x_n x_n)^{3/2}} \tag{56}$$

The numerator of the second fraction now reads $\delta_{jl}\delta_{ji}$. According to rule (20) this is the same as $\delta_{lj}\delta_{ji}$. According to rule (22) this contracts to δ_{li}:

$$v_i = \frac{3q_l\delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{li}}{\sqrt{x_n x_n}} - \frac{q_l x_m x_j \delta_{il}\delta_{jm}}{(x_n x_n)^{3/2}} + \frac{q_l x_m x_j \delta_{im}\delta_{jl}}{(x_n x_n)^{3/2}} \tag{57}$$

According to rule (23) the following can be simplified in the numerator of the third fraction: $x_m\delta_{jm} = x_j$.

$$v_i = \frac{3q_l\delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{li}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{il}x_j x_j}{(x_n x_n)^{3/2}} + \frac{q_l x_m x_j \delta_{im}\delta_{jl}}{(x_n x_n)^{3/2}} \tag{58}$$

Now, $\frac{x_j x_j}{(x_n x_n)^{3/2}}$ in the third fraction is nothing else than $\frac{1}{\sqrt{x_n x_n}}$:

$$v_i = \frac{3q_l\delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{li}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{il}}{\sqrt{x_n x_n}} + \frac{q_l x_m x_j \delta_{im}\delta_{jl}}{(x_n x_n)^{3/2}} \tag{59}$$

According to rule (20) the Kronecker-delta δ_{li} in the second fraction is the same as δ_{il}:

$$v_i = \frac{3q_l\delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{il}}{\sqrt{x_n x_n}} - \frac{q_l\delta_{il}}{\sqrt{x_n x_n}} + \frac{q_l x_m x_j \delta_{im}\delta_{jl}}{(x_n x_n)^{3/2}} \tag{60}$$

The first three fractions can now be combined:

$$v_i = \frac{q_l\delta_{il}}{\sqrt{x_n x_n}} + \frac{q_l x_m x_j \delta_{im}\delta_{jl}}{(x_n x_n)^{3/2}} \tag{61}$$

According to rule (23) the numerator of the first fraction can be further simplified as follows: $q_l\delta_{il} = q_i$

$$v_i = \frac{q_i}{\sqrt{x_n x_n}} + \frac{q_l x_m x_j \delta_{im}\delta_{jl}}{(x_n x_n)^{3/2}} \tag{62}$$

In the numerator of the second remaining fraction, according to rule (23), the following can be simplified: $x_m\delta_{im} = x_i$ and $x_j\delta_{jl} = x_l$:

$$v_i = \frac{q_i}{\sqrt{x_n x_n}} + \frac{(q_l x_l)x_i}{(x_n x_n)^{3/2}} \tag{63}$$

This can be back -converted into vector notation using rules (a), (b) and (d):

$$\boxed{\vec{v} = \frac{\vec{q}}{\|\vec{x}\|} + \frac{(\vec{q}\cdot\vec{x})\vec{x}}{\|\vec{x}\|^3}} \tag{64}$$

8 Appendix

When converting from index notation to vector-matrix notation, there are occasionally expressions for which the solution is not immediately obvious using the conversion rules given in chapter 4. For such cases, the following table offers a little help.

Index Notation	Equivalent Vector-Matrix-Notation
$\partial_i u_j v_j$	$\vec{\nabla}(\vec{u} \cdot \vec{v})$
$\partial_j u_i v_j$	$\left(\vec{\nabla}\vec{u}\right)^T \vec{v} \equiv \vec{\nabla}(\vec{u} \cdot \vec{v}) - \vec{v} \times \left(\vec{\nabla} \times \vec{u}\right)$
$\mathrm{M}_{ij} v_j$	$\underline{\underline{\mathrm{M}}}\vec{v}$
$v_i \mathrm{M}_{ij}$	$\vec{v}^T \underline{\underline{\mathrm{M}}} \equiv \vec{v} \cdot \underline{\underline{\mathrm{M}}}$
$\partial_i \mathrm{M}_{ij}$	$\vec{\nabla}^T \underline{\underline{\mathrm{M}}} \equiv \vec{\nabla} \cdot \underline{\underline{\mathrm{M}}}$
$\partial_j \mathrm{M}_{ij}$	$\left(\vec{\nabla}^T \underline{\underline{\mathrm{M}}}^T\right)^T \equiv \left(\vec{\nabla} \cdot \underline{\underline{\mathrm{M}}}^T\right)^T$
$u_j v_i w_j - u_j v_j w_i$	$\vec{u} \times \vec{v} \times \vec{w}$
$\partial_j u_i v_j - \partial_i u_j v_j$	$\left(\vec{\nabla} \times \vec{u}\right) \times \vec{v}$
$\partial_i \partial_j v_j - \partial_j \partial_j v_i$	$\vec{\nabla} \times \vec{\nabla} \times \vec{v}$

Note: The variants underlined in red ("inner product of a column vector with a matrix") may appear in the literature, but should be avoided, as there is no general consensus on this notation.